Persistence of information in the quantum measurement problem

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Abstract

The quantum measurement problem has to do with the compatibility of two different prescriptions for the end state of a quantum measurement process:
(1) a superposition representing an entangled state of object system and apparatus;
(2) a mixture of correlated states of object system and apparatus.
The first prescription is a consequence of the linearity of time evolution in quantum mechanics; the second one (corresponding to what is usually called the 'collapse of the wave function') is a basic feature of the axiomatic structure of the theory.
This paper proves that the two prescriptions are *exactly* equivalent whenever information about the results of the measurement persists, i.e. whenever there is some kind of record or trace of the results (all the measurements we are ordinarily concerned with fulfill this requirement). Experimental evidence is cited suggesting that indeed we should expect the equivalence of prescriptions (1) and (2) to hold *when and only when* information about measurement results persists. Finally the notion of 'element of reality' introduced by Einstein, Podolsky and Rosen is discussed in the context of this description of the measurement process, and a brief remark is offered about the 'classical' nature of the world we experience.

Résumé

Le problème quantique de la mésure concerne la compatibilité de deux différentes prescriptions pour l'état final d'un procès de mésure quantique:
(1) une superposition qui représente un état emmêlé du système observé et de l'appareil;
(2) un mélange d'états corrélés du système observé et de l'appareil.
La première prescription est une conséquence de la linéarité de l'évolution temporelle des états en mécanique quantique; la deuxième (correspondente à ce qu'on appèlle habituellement la 'réduction du paquet d'ondes') est un ingrédient essentiel de la structure axiomatique de la théorie quantique.
Dans ce travail je montre que ces deux prescriptions sont *exactement* équivalentes quand l'information sur les résultats de la mésure persiste, c'est-à-dire quand il y a un enregistrement ou une trace des résultats (toutes les observations qui nous intéressent normalement sont de ce genre).
Plusieurs experiments suggèrent qu'en effet on devrait s'attendre à que les prescriptions (1) et (2) soient équivalentes *si et seulement si* l'information sur les résultats de la mésure persiste. Finalement j'examine dans le contexte de cette description du procès de mésure le concept de 'élément de réalité' introduit par Einstein, Podolsky et Rosen et je propose quelques réflexions sur la nature 'classique' du monde que nous percevons.

Key-words: appearance of a classical world, collapse of the wave function, decoherence, element of reality, entanglement, information, quantum beats, quantum eraser, quantum measurement, von Neumann chain.

Introduction

The problem of measurement in quantum mechanics is a rare example of a fundamental problem at the heart of a very successful physical theory that has resisted three quarters of a century of investigation from many different points of view. In the context of this paper the problem will be defined as concerning the compatibility of two different prescriptions for the end state of a quantum measurement process:

(1) a superposition representing an entangled state of object system and apparatus;
(2) a mixture of correlated states of object system and apparatus corresponding to the various possible results of the measurement.

The first prescription is a consequence of the linearity of time evolution in quantum mechanics. It describes a quantum state in which the results of the measurement process are all simultaneously present and do not possess the ontological status of 'elements of reality.' Predictions based on this prescription contain interference terms mixing the various results. The second one (corresponding to what is usually called the 'collapse of the wave function') is a basic feature of the axiomatic structure of the theory as it is usually formulated and fits common sense and experimental evidence. In it the results of the measurement can be viewed as 'elements of reality' and predictions based on it treat the various measurement results as classical alternatives.

Historically the quantum measurement problem has been approached from many different angles. The approach proposed in this paper is based on the fact that the two prescriptions are *exactly* equivalent in all situations in which information about the results of the measurement persists, i.e. whenever some kind of record or trace of the results remains. This of course includes all the measurements we are ordinarily concerned with.

The meaning of the above statement can be made more precise in the following way. In general a measurement process involves a chain of interacting systems including, beside the object system and the apparatus, the laboratory environment, recording devices connected to the apparatus, possibly an observer reading the data, etc. In this chain of systems the measurement process causes changes which can be described in terms of the values assumed by 'measurement recording observables.' The claim of this paper is that

(a) prescription (1) is always correct, but
(b) prescriptions (1) and (2) are exactly equivalent for predicting the outcome of all observations that leave the value of at least one *measurement recording observable unaffected.*

There is no need to require the equivalence of (1) and (2) for predicting the outcome of observations that destroy all information about the measurement results (i.e. observations incompatible with *all* the measurement recording observables). In fact, there are good reasons to presume that in general the two prescriptions will *not* be equivalent, and that prescription (1) (the superposition), not prescription (2) (the mixture), should be used to correctly predict the outcome of such observations. Of course an experimental test of this is hard to realize when macroscopic measurement recording observables are involved, but some microscopic analogues strongly suggest what we should expect the outcome to be (see below, *Suggestive experimental evidence*).

The quantum measurement problem

An outline of a quantum measurement process can be easily sketched, following the treatment given by J. von Neumann (1932).[1] A measurement of the observable O on the quantum system $\mathcal{S}$ is an interaction between $\mathcal{S}$ and a system $\mathcal{M}$ (measuring apparatus) endowed with an observable M, which we shall call a 'measurement recording observable' and corresponds in an abstract sense to a 'pointer position.' The interaction is such that, if initially the observable O has a definite value $O = o_n$ in $\mathcal{S}$ and the measurement recording observable has the 'neutral' initial value $M = m_0$ in $\mathcal{M}$ ('zero pointer position'), at the end of the interaction $M = m_n$ in $\mathcal{M}$ ('the pointer has moved to the n-th position'). We shall additionally assume the measurement to be ideal, in the sense of creating a minimal disturbance on $\mathcal{S}$: this means that when the observable O has a definite value $O = o_n$ the state of $\mathcal{S}$ is left unchanged.

If U_t is the time evolution operator of the composite system $\mathcal{S} + \mathcal{M}$ during the interaction, the above description of a measurement process implies:

$$U_t \phi_n \otimes \Phi_0 = \phi_n \otimes \Phi_n, \tag{1}$$

where ϕ_n is an eigenstate of O corresponding to $O = o_n$ and Φ_n is an eigenstate of M corresponding to $M = m_n$. Therefore, by the linearity of U_t, if $\mathcal{S}$ is initially in a superposition of states corresponding to different values of O,

$$U_t \sum_n c_n \phi_n \otimes \Phi_0 = \sum_n c_n \phi_n \otimes \Phi_n, \tag{2}$$

i.e. the composite system $\mathcal{S} + \mathcal{M}$ ends up in a superposition of states corresponding to the various possible results. In the language of statistical mechanics, the system is represented by the density matrix (or statistical operator)

$$W_t = P_{\sum_n c_n \phi_n \otimes \Phi_n}. \tag{3}$$

On the other hand the usual axiomatic formulation of quantum mechanics requires the outcome of the measurement process to be such that in a fraction $|c_n|^2$ of the cases the object system is in the state ϕ_n and the measuring apparatus is in the state Φ_n, a situation described by the density matrix

$$\overline{W}_t = \sum_n |c_n|^2 P_{\phi_n \otimes \Phi_n}. \tag{4}$$

The quantum measurement problem has to do with the compatibility of the two prescriptions (3) and (4), the first a direct consequence of the linearity of time evolution, the second required by the axiomatic structure of the theory in accord with experimental evidence.

When we calculate the predictions of the density matrices (3) and (4) for subsequent observations performed on the composite system $\mathcal{S} + \mathcal{M}$, we can easily notice that they give the same expectation values for some observations, but not for all. The predictions coincide in particular for all measurements carried out only on the object system or only on the apparatus. But generally they do not for measurements involving correlations between the two systems. If we ask about the simultaneous probability of $\mathcal{S}$ possessing a property Q and $\mathcal{M}$ possessing a property R, in general the answers given by the density matrices (3) and (4) will be different.

We can think of Q and R as {0,1}-valued observables of $\mathcal{S}$ and $\mathcal{M}$ respectively, represented by projectors in the respective Hilbert spaces. If, e.g., Q = 1 when a certain observable A of $\mathcal{S}$ has a certain value a, and Q = 0 otherwise, then the corresponding operator Q is the projector on the eigenspace of A corresponding to the eigenvalue a, and its expectation value is the probability of A having the value a. Similarly, if R = 1 when the observable B of $\mathcal{M}$ has the value b, and R = 0 otherwise, the corresponding operator R is the projector on the eigenspace of B corresponding to the eigenvalue b, and its expectation value is the probability of the observable B having the value b. Then the expectation value $\langle Q \otimes R \rangle$ will be the probability of $\mathcal{S} + \mathcal{M}$ possessing both properties, i.e. having simultaneously A = a in $\mathcal{S}$ and B = b in $\mathcal{M}$. In the following we shall probe the statistics of the composite system $\mathcal{S} + \mathcal{M}$, or of its generalization (von Neumann chain) $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$; through expectation values of the form $\langle Q \otimes R \rangle$ ($\langle Q \otimes R^{(1)} \otimes R^{(2)} \otimes \ldots \otimes R^{(h)} \rangle$ for a von Neumann chain), where all the Q's and the R's are projectors. Knowledge of these expectation values fully specifies the statistics of the system.

If W is the density matrix of $\mathcal{S} + \mathcal{M}$, the expectation value $\langle Q \otimes R \rangle$ is given by

$$\left\langle Q \otimes R \right\rangle = Tr\left(WQ \otimes R \right). \qquad (5)$$

The density matrix (4) gives for this probability an expression of the form

$$\left\langle Q \otimes R \right\rangle = \sum_n \left| c_n \right|^2 \left\langle Q \otimes R \right\rangle_n , \qquad (6)$$

where $\langle Q \otimes R \rangle_n$ is the probability of both observables having the value 1 when we already know that the measurement of O on $\mathcal{S}$ has given the n-th result. In this expression the channels corresponding to different outcomes of the measurement are independent from each other: in a system described by the density matrix (4) establishing which result the measurement has given can be consistently interpreted as a mere increase of information, and the various results can be thought of as 'elements of reality' (in the sense of Einstein, Podolsky and Rosen[2]) for the composite system $\mathcal{S} + \mathcal{M}$.
The density matrix (3), on the other hand, gives for the expectation value (5) an expression of the form

$$\left\langle Q \otimes R \right\rangle = \sum_{n,n'} c_n c_{n'}^{\,*} \left\langle Q \otimes R \right\rangle_{n,n'} , \qquad (7)$$

which contains interference terms ($n \neq n'$) mixing the various channels. Thus the density matrix (3) describes the outcome of the measurement interaction as an entangled state of system and apparatus. In this case establishing which result the measurement has given cannot be interpreted as a mere increase of information – or, equivalently, we cannot claim that the measurement results are 'elements of reality.'

The quantum measurement problem is generally understood as the problem of either (i) describing a transition transforming (3) into (4) (collapse of the wave function); or (ii) proving that (3) and (4) are in some sense equivalent.

Two (rather different) examples of type (i) solutions are the 'idealistic' approach proposed by Wigner[3] in the 60's, in which the collapse of the wave function is an event reflecting the action of mind on matter; and the Ghirardi, Rimini and Weber[4] approach, which assumes that the wave function of a particle spontaneously collapses to a localized state with a very infrequent random process. A typical example of type (ii) solution is the 'decoherence' approach, in which the observable difference between the superposition and the mixture decays very rapidly in time due to the macroscopic nature of the measuring apparatus and of the environment in which the apparatus is immersed; superposition and mixture thus become approximately equivalent (with an approximation better than any conceivable experimental error), so that the first can be replaced by the second 'for all practical purposes.'[5]
The approach proposed in this paper is also of type (ii). It can be described in the following terms:

> *There is no collapse of the wave function and the superposition always correctly describes the outcome of a measurement process. But superposition and mixture are* exactly *equivalent for predicting the outcome of all subsequent observations that conserve information about the results of the measurement.*

Comments. The present proposal differs from the decoherence approach in two important respects. First, there is no approximation involved; second, the macroscopic nature of the systems involved plays no role in the equivalence proof. Persistence of information about the measurement results at any level, microscopic or macroscopic, is sufficient. E.g., in an ideal measurement the object system itself carries such information, which is only destroyed if the observation of a property Q incompatible with O is performed. On the other hand, if a measurement is immediately[6] followed by an observation erasing all information about its results (an observation incompatible with all the measurement recording observables) there is no reason to assume that the 'collapse of the wave function' axiom should be applicable. If the present approach is correct, the superposition (3) should correctly describe the outcome of such a measurement.

Suggestive experimental evidence

There is strong experimental evidence that, when a quantum process can take a number of different paths, the interference effects which are the signature of quantum superpositions only appear in the absence of information about which path the system has taken. The existence of any record of such information destroys the interference pattern and brings about a classical behavior. Two noteworthy examples are the 'quantum beats' experiments and the 'quantum eraser' experiments.

E.g., in the quantum beats experiment performed by Hellmuth, Walther, Zajonc and Schleich[7] two situations are compared. The first one involves 'type I' atoms, the relevant part of whose energy diagram consists of three levels: the ground level **g** and two close excited levels **a** and **a'** (generated by an energy level being split by the action of a weak magnetic field). The second situation involves 'type II' atoms, in whose energy diagram we also consider three levels, but this time it is the ground level that is split in **g** and **g'**, while the excited level **a** is not.

A beam of atoms is illuminated by pulsing laser light in such a way that some atoms get excited and subsequently decay emitting photons of the appropriate frequency: $\boldsymbol{\omega_{ga}}$ or $\boldsymbol{\omega_{ga'}}$ in the case of type I, $\boldsymbol{\omega_{ga}}$ or $\boldsymbol{\omega_{g'a}}$ in the case of type II. The detector is designed to accept one and only one photon for each laser pulse, so that a sequence of individual events is recorded. But the detector is insensitive to the photon's frequency, so that there is no distinction between pathways $\mathbf{a} \rightarrow \mathbf{g}$ and $\mathbf{a'} \rightarrow \mathbf{g}$ for type I atoms or between pathways $\mathbf{a} \rightarrow \mathbf{g}$ and $\mathbf{a} \rightarrow \mathbf{g'}$ for type II atoms. The number of events is plotted against decay time.

With type I atoms, superposed on the standard exponential decrease curve the experiment evidences the presence of an oscillatory phenomenon called 'quantum beats', analogous to the acoustic interference pattern of tuning forks whose frequencies are very close to each other. These oscillations are due to interference between the two possible pathways of the process: excitation to and decay from level **a** or **a.'** Since no track is kept of which-path information, a superposition describes the outcome of the experiment, and an interference pattern is observed.

The interesting twist is the fact that the outcome is completely different for type II atoms. Again there are two possible decay paths, $\mathbf{a} \rightarrow \mathbf{g}$ and $\mathbf{a} \rightarrow \mathbf{g'}$, and since we are collecting photons from both paths without distinguishing them, again we should expect interference to show up. But no quantum beats are observed in this case. Why? The difference is that now the final state of the atom records which one of the two processes has taken place. This is sufficient to erase the interference pattern. How should we understand this? Of course a superposition will still correctly describe the outcome of the experiment. But, according to the equivalence proof given in this paper, the superposition is exactly equivalent to the corresponding mixture with respect to all the observations that conserve which-path information (anywhere! at the microscopic or at the macroscopic level) and therefore no interference can be observed. Notice that it is irrelevant whether the experimenters have access to this information or not: just the information being available is enough.

The idea of a 'quantum eraser' was originally proposed around 1980 by Jaynes[8] and independently by Scully and Druhl,[9] and then it has since been realized in numerous experiments with a variety of setups.[10] Here it will be analyzed in the particularly clear form described by Greenstein and Zajonc,[11] based on the original suggestion by Scully and Druhl and the subsequent realization by Ou, Wang, Zou and Mandel.[12]

The basic setup is a Mach-Zehnder interferometer (FIG. 1). A beam of incoming photons is split into beams 1 and 2 by a first beam splitter BS_1, a phase shifter PS alters the phase of beam 2 by ϕ, and the two beams are re-combined by a second beam splitter BS_2. Detectors D_1 and D_2 reveal the outgoing photons. The incoming beam is of sufficiently low intensity, so that only one photon at a time is introduced in the apparatus. As the phase ϕ is varied, an interference pattern is observed in the counts of D_1 and D_2.

Now two non-linear down-conversion crystals, X_1, and X_2, are inserted on the photon's path. When the photon enters one of these crystals it is split into a pair: one member of the pair, which we shall call the primary photon, keeps on traveling through the interferometer, while the other, which we shall call the secondary photon, is detected by an additional counter, d_1 or d_2. The firing of d_1 or d_2 now provides which-path information about the primary photon, and the interference pattern disappears.

But the which-path information can be erased: if we place a third beam splitter BS_3 at the intersection of the paths of the secondary photon from the down-conversion crystals, the firing of d_1 or d_2 no longer tells us which path the primary photon has taken. The experiment then shows that an interference pattern re-appears in the coincidence counts between any of the primary detectors D_1 and D_2 and any of the secondary detectors d_1 and d_2.

We can look at the entanglement of the primary and secondary photon as a microscopic analogue of the entanglement of object system and macroscopic apparatus at the end of a measurement process, equation (2). The observed quantity O corresponds to a which-path observable of the primary photon and the measurement recording observable M corresponds to a which-path observable of the secondary photon. It can then be shown (see Appendix 1) that the interference pattern shows up when and only when we perform a joint observation on the entangled pair that is incompatible with both which-path observables, i.e. when we measure a quantity of the form $Q \otimes R$, with $[O, Q] \neq 0$ and $[M, R] \neq 0$. Such an observation erases all information about the previous 'measurement': in these circumstances the equivalence of the superposition (3) and the mixture (4) does not hold, and we indeed expect an interference effect to show up.

It is interesting to notice that the quantum eraser can be performed as a 'delayed choice' experiment and it has in fact been performed as such.[13] I.e., in principle one could insert the beam splitter BS_3 after the photon has already interacted with BS_1, and even after it has already interacted with X_1 and X_2, thereby forcing it to choose between "going both ways" and "going either one way or the other " when it is already "well on its way." This peculiar feature of quantum mechanics has been the object of some discussion, and some people feel that it implies in some way "changing the past." No such doubt can arise in the approach of the present paper, in which inserting or not inserting the third beam splitter simply corresponds to performing different observations on the system. If the observation we choose to perform conserves information about the previous 'measurement ', the equivalence proof will hold and an "either-or" situation will appear; otherwise it will not.

Unless we believe that there is a fundamental divide between microscopic and macroscopic physics, it is difficult to imagine that we would not obtain the same results if we could perform a quantum eraser experiment on a macroscopic scale. I.e., if we could carry out a measurement and immediately after perform on the von Neumann chain of recording systems involved an observation incompatible with all the measurement recording observables, so as not to leave any trace of the previous measurement results (no trace anywhere, not even in the environment!), we should expect the outcome of such an observation to be correctly predicted by the superposition (3), not by the mixture (4). The experiment would thus explicitly demonstrate that there is properly no such thing as a 'collapse of the state vector' in a quantum measurement.

Proof of the equivalence statement

As previously stated, a measurement process involves more than a microsystem and an apparatus: the apparatus is constantly interacting with an environment, and correlations are quickly established between the state of the apparatus and that of the environment, as decoherence research has shown. In addition to that, we are usually interested in measurements whose results are recorded in some kind of device or in the brain of a human observer or both. An actual measurement process is therefore better described as an interaction involving a 'von Neumann chain' of systems $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$, each one of which records the outcome of the measurement in the value of an observable O, $M^{(1)}$, $M^{(2)}$, …, $M^{(h)}$ (we call these 'measurement recording observables').

E.g., in a measurement involving photons, the photon may interact with a photomultiplier, the output of the photomultiplier may be sent to an amplifier, the output of the amplifier may be sent to a counter, the output of the counter may be sent to a computer and finally an experimenter may read the computer screen. In all these systems correlated changes take place: after interaction the state of the photomultiplier is correlated with the state of the photon, the state of the amplifier with that of the photomultiplier, the state of the counter with that of the amplifier, the state of the computer with that of the counter, and the state of the experimenter's retina, optic nerve, brain, etc. with that of the computer. But the von Neumann chain involved in the measurement could extend further: all changes happening at a macroscopic level affect the environment. E.g., different states of the computer may correspond to different states of illumination of its screen, which in turn may correspond to different states of illumination of the laboratory environment, etc. Given sufficient time, the von Neumann chain of a quantum measurement spreads through the universe. Fortunately in the following we don't need to specify which systems are included in the von Neumann chain. But it is important to keep in mind this extended meaning of the von Neumann chain because, if the condition for the equivalence of the superposition (3) and the mixture (4) is that a record of the measurement results is conserved, this record can be anywhere along the chain, including, e.g., in the environment.

The observations that conserve a record of the measurement results are those that leave the value of at least one of the measurement recording observables unperturbed. An observation that destroys all records of the previous measurement performed on $\mathcal{S}$, on the other hand, consists of the simultaneous measurement of observables Q, $R^{(1)}$, $R^{(2)}$, …, $R^{(h)}$ such that $[Q,O] \neq 0$, $[R^{(1)}, M^{(1)}] \neq 0$, …, $[R^{(h)}, M^{(h)}] \neq 0$.

A sufficiently general proof of the equivalence statement requires us to extend the formalism of equations (1)-(6) in two ways. First, we need replace the composite system $\mathcal{S} + \mathcal{M}$ with a chain of interacting systems $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$; and second, we need to allow for degeneracy in the measurement recording observables. In order to keep our notation simple, we shall perform these two extensions separately. First we shall prove the equivalence of superposition and mixture for the simpler system $\mathcal{S} + \mathcal{M}$, allowing the measurement recording observable M to be degenerate (for the sake of simplicity we shall not consider degeneracy in the object system observable O, although that is a straightforward generalization). Then we shall give the same proof for the chain of systems $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$ in the non degenerate case. How to deal with the general case of a chain of systems with degenerate observables will then be perfectly obvious, although cumbersome because of the many indices involved.

Let $\{\phi_n\}$ be a complete orthonormal set of eigenstates of O in $\mathcal{H}^{(S)}$, the Hilbert space of $\mathcal{S}$, and $\{\Phi_{nk}\}$ a complete orthonormal set of eigenstates of M in $\mathcal{H}^{(M)}$, the Hilbert space of $\mathcal{M}$:

$$O\phi_n = o_n\phi_n, \quad M\Phi_{n,k} = m_n\Phi_{n,k}. \tag{8}$$

If the observed system $\mathcal{S}$ is initially in the state

$$\phi = \sum_n c_n\phi_n, \tag{9}$$

its density matrix in $\mathcal{H}^{(S)}$ is

$$W_0^{(S)} = P_{\sum_n c_n\phi_n}. \tag{10}$$

The initial quantum state of the measuring apparatus will not be exactly known: we can only assume that it will be a mixture (with weights w_k unknown) of states corresponding to the 'neutral pointer position',

$$W_0^{(M)} = \sum_k w_k P_{\Phi_{0,k}} . \tag{11}$$

The composite system $\mathcal{S} + \mathcal{M}$ will then be represented by the density matrix

$$W_0 = \sum_k w_k P_{\sum_n c_n \phi_n} \otimes P_{\Phi_{0,k}} = \sum_k w_k P_{\sum_n c_n \phi_n \otimes \Phi_{0,k}} \tag{12}$$

in the tensor product Hilbert space $\mathcal{H} = \mathcal{H}^{(S)} \otimes \mathcal{H}^{(M)}$ (where we have used the fact that, when the component systems are in pure states ϕ and Φ respectively, the composite system is in the pure state ϕ⊗Φ, see Appendix 2). (In order to keep our notation simple we shall not explicitly indicate which Hilbert space a projector acts in and rely on the context to make it clear. E.g., in equation (12) $P_{\sum_n c_n \phi_n}$ acts in $\mathcal{H}^{(S)}$, $P_{\Phi_{0,k}}$ acts in $\mathcal{H}^{(M)}$ and $P_{\sum_n c_n \phi_n \otimes \Phi_{0,k}}$ acts in $\mathcal{H}^{(S)} \otimes \mathcal{H}^{(M)}$.)

The time evolution of the density matrix of the composite system is governed by the Liouville-von Neumann equation

$$\tfrac{d}{dt} W_t = -i[H, W_t], \tag{13}$$

where H is the total Hamiltonian of $\mathcal{S} + \mathcal{M}$. The solution of equation (13) can be written in the form

$$W_t = \mathcal{U}_t W_0 = U_t W_0 U_t^{-1}, \tag{14}$$

where U_t is the unitary time evolution operator of $\mathcal{S} + \mathcal{M}$, which, in the case of a measurement process, must fulfill the requirement of equation (1). Allowing for degeneracy in M we may rewrite equation (1) as

$$U_t \phi_n \otimes \Phi_{0,k} = \phi_n \otimes \Phi_n(k), \tag{15}$$

where Φ_n (k) is an eigenstate of M corresponding to M = m_n, in general dependent upon the initial state of the apparatus Φ_{0k}.

Applying time evolution to the initial density matrix (12) we obtain

$$\begin{aligned} W_t &= \sum_k w_k \, \mathcal{U}_t \, P_{\sum_n c_n \phi_n \otimes \Phi_{0,k}} \\ &= \sum_k w_k P_{U_t \sum_n c_n \phi_n \otimes \Phi_{0,k}} \\ &= \sum_k w_k P_{\sum_n c_n \phi_n \otimes \Phi_n(k)}, \end{aligned} \tag{16}$$

where we have made use of the fact that $\mathcal{U}_t \, P_\psi = P_{U_t \psi}$ (see Appendix 3).

Equation (16) describes the outcome of the measurement interaction and generalizes equation (3). Let us now evaluate the expectation value of a joint property Q⊗R of $\mathcal{S} + \mathcal{M}$ with the density matrix (16):

$$\langle Q \otimes R \rangle = Tr(W_t Q \otimes R). \tag{17}$$

If $\{\psi_i\}$ is a complete orthonormal set in $\mathcal{H}^{(S)}$ and $\{\Psi_j\}$ a complete orthonormal set in $\mathcal{H}^{(M)}$, the trace in equation (17) can be calculated as follows:

$$Tr(W_t Q \otimes R)$$

$$=\sum_{i,j}\left(\psi_i\otimes\Psi_j, W_t Q\otimes R\psi_i\otimes\Psi_j\right)$$

$$=\sum_{i,j,k} w_k\left(\psi_i\otimes\Psi_j, P_{\sum_n c_n\phi_n\otimes\Phi_n(k)} Q\otimes R\psi_i\otimes\Psi_j\right)$$

$$=\sum_{i.j.k} w_k\left(\psi_i\otimes\Psi_j, \sum_n c_n\phi_n\otimes\Phi_n(k)\right)\left(\sum_{n'} c_{n'}\phi_{n'}\otimes\Phi_{n'}(k), Q\otimes R\psi_i\otimes\Psi_j\right)$$

$$=\sum_{i,j,k,n,n'} c_n c_{n'}^{*} w_k(\psi_i,\phi_n)(\Psi_j,\Phi_n(k))(\phi_{n'},Q\psi_i)(\Phi_{n'}(k),R\Psi_j), \qquad (18)$$

where we have made use of the definition of the scalar product in the composite space

$$(\phi\otimes\Phi,\psi\otimes\Psi)=(\phi,\psi)(\Phi,\Psi). \qquad (19)$$

Now, if R commutes with M, $\{\Psi_j\}$ can be chosen to be a complete orthonormal set of eigenfunctions common to both observables. Replacing the index j with j, j', j":

$$M\Psi_{j,j',j''}=m_j\Psi_{j,j',j''}, \qquad R\Psi_{j,j',j''}=r_{j'}\Psi_{j,j',j''}, \qquad (20)$$

and

$$Tr(W_t Q\otimes R)$$

$$=\sum_{i,j,j',j'',k,n,n'} c_n\, c_{n'}^{*} w_k(\psi_i,\phi_n)(\Psi_{j,j',j''},\Phi_n(k))(\phi_{n'},Q\psi_i)(\Phi_{n'}(k),R\Psi_{j,j',j''})$$

$$=\sum_{i,j,j',j'',k,n,n'} c_n\, c_{n'}^{*} w_k r_{j'}(\psi_i,\phi_n)(\Psi_{j,j',j''},\Phi_n(k))(\phi_{n'},Q\psi_i)(\Phi_{n'}(k),\Psi_{j,j',j''}) \qquad (21)$$

Now, since $\Phi_n(k)$ is an eigenstate of M corresponding to the eigenvalue m_n, $\Phi_{n'}(k)$ is an eigenstate of M corresponding to the eigenvalue $m_{n'}$ and $\Psi_{j,j',j''}$ is an eigenstate of M corresponding to the eigenvalue m_j, the product

$$(\Psi_{j,j',j''},\Phi_n(k))(\Phi_{n'}(k),\Psi_{j,j',j''}) \qquad (22)$$

vanishes unless both j = n and j = n', which implies n' = n and

$$\langle Q\otimes R\rangle = Tr(W_t Q\otimes R)$$

$$=\sum_{i,j',j'',k,n} |c_n|^2 w_k r_{j'}(\psi_i,\phi_n)(\Psi_{n,j',j''},\Phi_n(k))(\phi_n,Q\psi_i)(\Phi_n(k),\Psi_{n,j',j''})$$

$$=\sum_{i,j',j'',k,n} |c_n|^2 w_k(\psi_i,\phi_n)(\Psi_{n,j',j''},\Phi_n(k))(\phi_n,Q\psi_i)(\Phi_n(k),R\Psi_{n,j',j''})$$

$$=\sum_{k,n}|c_n|^2 w_k Tr\left(P_{\phi_n\otimes\Phi_n(k)}Q\otimes R\right)$$

$$=Tr\left(\overline{W}_t Q\otimes R\right), \qquad (23)$$

where

$$\overline{W_t} = \sum_{k,n} \left|c_n\right|^2 w_k P_{\phi_n \otimes \Phi_n(k)}, \tag{24}$$

which is the form taken by the density matrix (4) when degeneracy is taken into account. Equation (23) establishes the equivalence of the density matrices (16) and (24) for predicting the outcome of joint observations of the type Q⊗R on the composite system $\mathcal{S} + \mathcal{M}$ when R commutes with M. The same argument obviously applies when Q commutes with O: therefore we conclude that the density matrices (16) and (24) are exactly equivalent for predicting the outcome of joint observations on the composite system $\mathcal{S} + \mathcal{M}$ when at least one of the observables commutes with the corresponding measurement recording observable.

The generalization of this result to a von Neumann chain of recording systems is straightforward. Ignoring degeneracy, as previously stated, in order to keep the notation simple, the initial state of the system $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$ is described by the density matrix

$$W_0 = P_{\sum_n c_n \phi_n \otimes \Phi_0^{(1)} \otimes \ldots \otimes \Phi_0^{(h)}}, \tag{25}$$

where $\Phi_0^{(k)}$ is the initial state of $\mathcal{M}^{(k)}$, k = 1, …, h. The measurement interaction carries this density matrix into

$$W_t = \mathcal{U}_t W_0 = P_{\sum_n c_n \phi_n \otimes \Phi_n^{(1)} \otimes \ldots \otimes \Phi_n^{(h)}}. \tag{26}$$

The expectation value of a joint observation of the properties Q, $R^{(1)}$, $R^{(2)}$,…, $R^{(h)}$ on the system $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$ is

$$\left\langle Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)} \right\rangle = Tr\left(W_t Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)}\right) \tag{27}$$

If $\{\psi_i\}$ is a complete orthonormal set in $\mathcal{H}^{(S)}$ and $\{\Psi_{j_k}^{(k)}\}$ a complete orthonormal set in $\mathcal{H}^{(k)}$, the above trace can be evaluated as

$$Tr\left(W_t Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)}\right)$$
$$= \sum_{i, j_1, \ldots, j_h} \left(\psi_i \otimes \Psi_{j_1}^{(1)} \otimes \ldots \otimes \Psi_{j_h}^{(h)}, P_{\sum_n c_n \phi_n \otimes \Phi_n^{(1)} \otimes \ldots \otimes \Phi_n^{(h)}} Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)} \psi_i \otimes \Psi_{j_1}^{(1)} \otimes \ldots \otimes \Psi_{j_h}^{(h)} \right)$$
$$= \sum_{i, j_1, \ldots, j_h, n, n'} c_n\, c_{n'}^{*} (\psi_i, \phi_n)\left(\Psi_{j_1}^{(1)}, \Phi_n^{(1)}\right) \ldots \left(\Psi_{j_h}^{(h)}, \Phi_n^{(h)}\right)\left(\phi_{n'}, Q\psi_i\right)\left(\Phi_{n'}^{(1)}, R^{(1)}\Psi_{j_1}^{(1)}\right) \ldots \left(\Phi_{n'}^{(h)}, R^{(h)}\Psi_{j_h}^{(h)}\right) \tag{28}$$

Now the same argument we used for equation (18) applies to this expression. If one of the R's commutes with the corresponding M, say $[R^{(1)}, M^{(1)}] = 0$, we can choose { $\Psi_{j_1}^{(1)}$ } as a complete orthonormal set of eigenfunctions common to both observables. As a consequence the product

$$\left(\Psi_{j_1}^{(1)}, \Phi_n^{(1)}\right)\left(\Phi_{n'}^{(1)}, R^{(1)}\Psi_{j_1}^{(1)}\right) \tag{29}$$

vanishes unless both $j_1 = n$ and $j_1 = n'$, which implies n' = n and

$$Tr\left(W_t Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)}\right)$$
$$= \sum_{i, j_2, \ldots, j_h, n} \left|c_n\right|^2 (\psi_i, \phi_n)\left(\Psi_n^{(1)}, \Phi_n^{(1)}\right) \ldots \left(\Psi_{j_h}^{(h)}, \Phi_n^{(h)}\right)\left(\phi_n, Q\psi_i\right)\left(\Phi_n^{(1)}, R^{(1)}\Psi_n^{(1)}\right) \ldots \left(\Phi_n^{(h)}, R^{(h)}\Psi_{j_h}^{(h)}\right)$$
$$= Tr\left(\overline{W_t} Q \otimes R^{(1)} \otimes \ldots \otimes R^{(h)}\right) \tag{30}$$

where

$$\overline{W_t} = \sum_n |c_n|^2 P_{\phi_n \otimes \Phi_n^{(1)} \otimes \ldots \otimes \Phi_n^{(h)}} . \tag{31}$$

We obviously reach the same conclusion if Q commutes with O; therefore the density matrices (26) and (31) are (exactly) equivalent for predicting the outcome of joint observations on the composite system $\mathcal{S} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \ldots + \mathcal{M}^{(h)}$ when at least one of the observables Q, $R^{(1)}$, $R^{(2)}$,…, $R^{(h)}$ commutes with the corresponding measurement recording observable O, $M^{(1)}$, $M^{(2)}$, … or $M^{(h)}$.

'Elements of reality'

In their 1935 paper[2] Einstein, Podolsky and Rosen gave the following definition of an 'element of reality':

> "If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

Equivalently we can say that a property of a physical system corresponds to an 'element of reality' if ascertaining it can be regarded as a mere increase in information. In classical physics all the properties of a physical system correspond to 'elements of reality.'
The notion of 'element of reality' is much more problematic in quantum physics (which is why Einstein, Podolsky and Rosen felt the need to introduce the concept in the first place!) and the difficulty is essentially connected with the description of the measurement process.

In the language of statistics a 'mere increase in information' corresponds to the reduction of an ensemble to a sub-ensemble. E.g., let us assume that we have a set of identical systems which can be divided into subsets labeled by the index n. In the n-th subset (comprising a fraction $|c_n|^2$ of the ensemble) the system is in the state ϕ_n with certainty, and the states ϕ_n are mutually exclusive (orthogonal vectors in a Hilbert space in the language of quantum physics). In quantum physics this situation is represented by the density matrix

$$W = \sum_n |c_n|^2 P_{\phi_n} \tag{32}$$

Ascertaining whether a given system possesses the property of being in the state ϕ_n does not imply "in any way disturbing the system": it only requires determining to which subset the system belongs. This corresponds to reducing the density matrix (32) to a single one of its terms and can be regarded as a mere increase in information. Therefore, according to the definition given above, we can say that in this context the property "the system is in the state ϕ_n" corresponds to an 'element of reality.'

But it is important to notice that, while the notion of 'element of reality' is perfectly unambiguous in classical physics, it is context-dependent in quantum physics. E.g., if a system $\mathcal{S}$ consists of sub-systems $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, with Hilbert spaces $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, both the density matrices

$$\sum_n |c_n|^2 P_{\phi_n^{(1)} \otimes \phi_n^{(2)}} \quad \text{and} \quad P_{\sum_n c_n \phi_n^{(1)} \otimes \phi_n^{(2)}} \tag{33}$$

when projected on $\mathcal{H}^{(1)}$ give

$$\sum_n |c_n|^2 P_{\phi_n^{(1)}} . \tag{34}$$

But the density matrices (33) are profoundly different, as we have seen above, and, while in the situation described by the first one the occurrence of the state $\phi_n^{(1)} \otimes \phi_n^{(2)}$ can be regarded as an 'element of reality' for the system $\mathcal{S}$, in the situation described by the second it cannot. In general, the density matrix of a composite system $\mathcal{S}$ determines those of its sub-systems $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, but not the other way around: i.e., if a certain property is an 'element of reality' for the composite system $\mathcal{S}$, there will certainly be a corresponding 'element of reality' in each one of its sub-systems $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$; but not necessarily the other way around. In quantum physics an 'element of reality' on a certain scale does not need to be one on a larger scale.

It is then useful to introduce the following context-dependent notion of 'element of reality': *the value of the observable* M *with spectral decomposition*

$$M = \sum_l m_l P_l \tag{35}$$

is an 'element of reality' for the quantum system $\mathcal{S}$ described by the density matrix W *with respect to the set of observables* {T} *if*

$$Tr(WT) = Tr\left(W^{(M)}T\right), \quad \forall T \in \{T\} \tag{36}$$

where

$$W^{(M)} = \sum_l P_l W P_l \tag{37}$$

It is easy to verify, on the ground of the arguments previously given, that:
($\mathcal{R}$) *the result of a quantum measurement is an 'element of reality' with respect to all the observables that commute with at least one measurement recording observable* (see Appendix 4). On the other hand, the result of a quantum measurement is not an 'element of reality' with respect to those observables that do not commute with any of the measurement recording observables.

One way of looking at this is the following. The correlation requirement (1) by itself is an incomplete definition of a quantum measurement process: in order to have a 'proper measurement' we have to add the condition that some record of that correlation persists. If we accept this last definition, we can simply state the thesis of this paper as: *in quantum physics the result of a proper measurement is an 'element of reality.'*

A final remark. Classical physics does not need to deal explicitly with the mind-matter interface problem because it regards all properties of a physical system as 'elements of reality.' In quantum physics, on the other hand, only those properties of a physical system about which information is stored somewhere can be regarded as 'elements of reality.' 'Somewhere' can be, e.g., in a human "brain."[14] It has been claimed that the observer plays an essential role in the quantum measurement process, and that the world appears 'classical' to us (i.e. it appears to consist of 'elements of reality') because the observer's consciousness 'collapses the wave function.' From the point of view of the present paper the observer does indeed play a role in the classical appearance of the world, but for a reason which has nothing to do with the collapse of the wave function. The world appears 'classical' to us because we can only deal with information that gets recorded in some observer's brain: in all the measurements we come to know about, one of the measurement recording observables corresponds to the state of an observer's brain.

This can be described (very schematically) in the language of von Neumann chains of recording systems in the following way. Let us assume that we are talking about a measurement whose results come to be known by an observer. That means that there will be at least one observer's brain in the von Neumann chain of recording systems. Let us say that $\mathcal{B}^{(1)}$ is such an observer's brain, and $M^{(\mathcal{B}(1))}$ is the corresponding measurement recording observable. Then $M^{(\mathcal{B}(1))} = m^{(\mathcal{B}(1))}{}_n$ corresponds to the statement "the observer is aware that the measurement has given the nth result." Now let us enquire about the probabilities for a subsequent joint observation of the properties $Q \otimes R^{(1)} \otimes R^{(2)} \otimes \ldots \otimes R^{(\mathcal{B}(1))} \otimes \ldots \otimes R^{(h)}$, performed on any or all of the systems in the von Neumann chain including the observer's brain $\mathcal{B}^{(1)}$. We have two possibilities:

(i) either $R^{(\mathcal{B}(1))}$ commutes with $M^{(\mathcal{B}(1))}$ and does not erase the observer's awareness of the measurement having given the n^{th} result (we may indeed wish that $R^{(\mathcal{B}(1))} = I^{(\mathcal{B}(1))}$, the identity in the Hilbert space of $\mathcal{B}^{(1)}$, meaning that the observer's brain is left alone!);
(ii) or, bad luck for our observer, $R^{(\mathcal{B}(1))}$ is a quantum measurement performed on her or his brain, and moreover $R^{(\mathcal{B}(1))}$ does not commute with $M^{(\mathcal{B}(1))}$, so that the awareness of the measurement having given the n^{th} result gets erased.
In the first case, since one of the R's commutes with the corresponding M, the equivalence condition is satisfied, information about the measurement results survives in the observer's brain and all predictions given by the superposition (26) are identical to those given by the mixture (31). The measurement results can be regarded as 'elements of reality.'
In the second case, since we are assuming that the measurement results eventually come to be known by an observer, there must be at least another observer's brain $\mathcal{B}^{(2)}$ in the von Neumann chain, and we can reason in the same way about $\mathcal{B}^{(2)}$. And so on. Eventually we have to reach an observer's brain whose awareness of the measurement having given the n^{th} result does not get erased. Then again the equivalence condition is satisfied, all predictions given by the superposition (26) are identical to those given by the mixture (31) and the measurement results can be regarded as 'elements of reality.'

Summarizing, for all measurements whose results eventually come to be known by some observer the superposition (26) is (exactly!) indistinguishable from the corresponding mixture (31). We see the occurrence of various measurement results as "either-or", i.e. as 'elements of reality', in spite of the fact that quantum mechanics tells us they are "and-and." The quantum nature of the world eludes our direct perception, not because it is not there on the macroscopic level, but because we are "part of the von Neumann chain" observing the world.

About those measurements whose results never come to be known by any observer we cannot say anything *a priori* (but notice that the equivalence statement, and therefore the possibility to talk about the measurement results as 'elements of reality', still holds if a record of those results persists anywhere, even if nobody ever reads it!).

Our image of the world, of course, is built on information that we do come to know, that does get recorded in some observer's brain. We do not stand apart from the world we observe: observing means causing a trace to appear in the world, and our knowing is inescapably conditioned by that. That is why the world appears 'classical' to us.

Appendix 1
The quantum eraser

A highly simplified model of a quantum eraser experiment is here described in order to emphasize its essential quantum features and highlight the close analogy with the treatment of the quantum measurement process given in the present paper. For the quantum optical calculations and the actual experimental setup the reader is referred to the paper by Ou, Wang, Zou and Mandel.[15] The experiment (see FIG.1) will be described in three steps.

Step 1. A photon enters a Mach-Zehnder interferometer, is split by a beam-splitter BS_1 and travels on along path 1 or path 2. A phase shifter PS alters the phase of the partial wave traveling along path 2 by ϕ. We shall represent the state of the photon as a vector in a two-dimensional Hilbert space $\mathcal{H}^{(S)}$ spanned by the complete orthonormal set of states $\{|1\rangle, |2\rangle\}$, where $|1\rangle$ corresponds to the photon traveling along path 1 and $|2\rangle$ corresponds to the photon traveling along path 2. After interaction with BS_1 and PS the state of the photon can then be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + e^{i\phi}|2\rangle\right). \qquad \text{(A1.1)}$$

When we include the beam splitter BS_2 we are brought to consider another complete orthonormal set of states in $\mathcal{H}^{(S)}$, $\{|1'\rangle, |2'\rangle\}$, where $|1'\rangle$ describes a photon traveling past BS_2 toward the detector D_1, and $|2'\rangle$ describes a photon traveling past BS_2 toward the detector D_2. The transformation between the two orthonormal sets is determined by the interaction with the beam splitter, schematically described in FIG. 2:

$$\begin{aligned} |1\rangle &= \frac{1}{\sqrt{2}}\left(|1'\rangle - |2'\rangle\right) \\ |2\rangle &= \frac{1}{\sqrt{2}}\left(|1'\rangle + |2'\rangle\right) \end{aligned}, \qquad \text{(A1.2)}$$

In terms of $|1'\rangle$ and $|2'\rangle$ the state of the photon can then be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\left(1 + e^{i\phi}\right)|1'\rangle - \left(1 - e^{i\phi}\right)|2'\rangle\right]. \qquad \text{(A1.3)}$$

Let us now define 'detector observables' for the counters D_1 and D_2. If there is no beam splitter BS_2, these are the projectors

$$\begin{aligned} D_1 &= |1\rangle\langle 1| \\ D_2 &= |2\rangle\langle 2| \end{aligned}. \qquad \text{(A1.4)}$$

$\mathbf{D_i} = 1$ if the photon travels along path i (i = 1,2) and is detected by the counter Di, $\mathbf{D_i} = 0$ otherwise. When the beam splitter BS_2 is in place, the 'detector observables' for the counters D_1 and D_2 become

$$\begin{aligned} D'_1 &= |1'\rangle\langle 1'| \\ D'_2 &= |2'\rangle\langle 2'| \end{aligned}. \qquad \text{(A1.5)}$$

We are now in a position to calculate the probability of the photon being detected by counter D_1, respectively D_2:

$$\begin{aligned} \langle\psi'|D'_1|\psi'\rangle &= \tfrac{1}{4}\left(1 + e^{-i\phi}\right)\left(1 + e^{i\phi}\right) = \tfrac{1}{2}\left(1 + \cos\phi\right) = \cos^2 \tfrac{\phi}{2} \\ \langle\psi'|D'_2|\psi'\rangle &= \tfrac{1}{4}\left(1 - e^{-i\phi}\right)\left(1 - e^{i\phi}\right) = \tfrac{1}{2}\left(1 - \cos\phi\right) = \sin^2 \tfrac{\phi}{2} \end{aligned}. \qquad \text{(A1.6)}$$

Since the interaction with BS_2 destroys which-path information, as expected, varying the phase shift ϕ we obtain an interference pattern.

Step 2. Let us now insert two identical non-linear down-conversion crystals X_1 and X_2 on the photon's path, and place additional detectors d_1 and d_2 in such a way as to detect the secondary photon generated by X_1 or X_2. We shall describe the state of the secondary photon as a vector in a two-dimensional Hilbert space $\mathcal{H}^{(\mathcal{M})}$ spanned by the complete orthonormal set of states $\{| x_1 \rangle, | x_2 \rangle\}$, where $| x_1 \rangle$ corresponds to the photon traveling along path x_1 toward the detector d_1, and $| x_2 \rangle$ corresponds to the photon traveling along path x_2 toward the detector d_2. Obviously the 'detector observables' for the counters d_1 and d_2 will be

$$d_1 = |x_1\rangle\langle x_1| \qquad d_2 = |x_2\rangle\langle x_2| . \tag{A1.7}$$

The composite system of primary and secondary photon will be described by a vector in a four-dimensional Hilbert space $\mathcal{H} = \mathcal{H}^{(S)} \otimes \mathcal{H}^{(\mathcal{M})}$. To keep our notation simple we shall still denote the state of the primary photon traveling along path 1 as $| 1 \rangle$ and the state of the photon traveling along path 2 as $| 2 \rangle$. Then the state of the composite system will be:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle|x_1\rangle + e^{i\phi}|2\rangle|x_2\rangle\right), \tag{A1.8}$$

an entangled state analogous to the superposition (2) resulting from the interaction of an object system with a measuring apparatus.

We can therefore look upon this process as a microscopic analogue of a measurement. The primary photon is the observed system $\mathcal{S}$, and one of its which-path observables, $\mathbf{D_1}$ or $\mathbf{D_2}$ (which are of course equivalent, since $\mathbf{D_1} = 1$ implies $\mathbf{D_2} = 0$ and viceversa) plays the role of the measured quantity O. The secondary photon corresponds to the measuring apparatus $\mathcal{M}$, and one of its which-path observables, $\mathbf{d_1}$ or $\mathbf{d_2}$, can be taken as the 'measurement recording observable' M. Equation (A1.8) establishes a correlation between the value of O and the value of M.

When we include the beam splitter BS_2 we can again write the states $| 1 \rangle$ and $| 2 \rangle$ in terms of a new pair, which again we shall denote as $| 1' \rangle$ and $| 2' \rangle$, according to equation (A1.2), and the state of the entangled pair (A1.8) becomes

$$|\psi\rangle = \frac{1}{2}\left[|1'\rangle\left(|x_1\rangle + e^{i\phi}|x_2\rangle\right) - |2'\rangle\left(|x_1\rangle - e^{i\phi}|x_2\rangle\right)\right]. \tag{A1.9}$$

We can now calculate the probabilities for separate detection of one or the other photon as expectation values of quantities of the form $\mathbf{D'_i} \otimes \mathbf{I_x}$ or $\mathbf{I} \otimes \mathbf{d_j}$, where $\mathbf{I}$ is the identity operator in $\mathcal{H}^{(S)}$ and $\mathbf{I_x}$ is the identity operator in $\mathcal{H}^{(\mathcal{M})}$, and the probabilities for coincidences of a primary and a secondary detector as expectation values of quantities of the form $\mathbf{D'_i} \otimes \mathbf{d_j}$. We obtain:

$$\begin{aligned} \langle\psi|D'_i \otimes d_j|\psi\rangle &= \tfrac{1}{4} \\ \langle\psi|D'_i \otimes I_x|\psi\rangle &= \langle\psi|I \otimes d_j|\psi\rangle = \tfrac{1}{2} \end{aligned} \qquad (i,j = 1,2). \tag{A1.10}$$

None of these exhibits an interference pattern, which is not surprising, since which-path information is here conserved. (Notice that it suffices that the information is recorded in the state of the secondary photon, even when no measurement is performed on it.) In 'measurement process language' we shall simply notice that we are here observing quantities of the form Q⊗R, with R commuting with M (since the **d** observables obviously commute with themselves and with the identity). Therefore the equivalence condition is satisfied, and the superposition (A1.8) gives the same results as the corresponding mixture, i.e. the calculation can be performed taking each term of (A1.8) as representing the state of a separate sub-ensemble.

Step 3. We now insert a third beam splitter BS_3 at the intersection of the two paths of the secondary photon. In complete analogy to what we did for BS_2, we can write the states of the secondary photon in terms of a new complete orthonormal set of states in $\mathcal{H}^{(\mathcal{M})}$, $\{|x'_1\rangle, | x'_2\rangle\}$, where |

x'$_1\rangle$ describes a photon traveling past BS_3 toward the detector d_1, and $|x'_2\rangle$ describes a photon traveling past BS_3 toward the detector d_2:

$$|x_1\rangle = \frac{1}{\sqrt{2}}\left(|x'_1\rangle - |x'_2\rangle\right)$$
$$|x_2\rangle = \frac{1}{\sqrt{2}}\left(|x'_1\rangle + |x'_2\rangle\right). \qquad \text{(A1.11)}$$

The 'detector observables' for counters d_1 and d_2 will now be

$$\mathbf{d'}_1 = |x'_1\rangle\langle x'_1|$$
$$\mathbf{d'}_2 = |x'_2\rangle\langle x'_2|, \qquad \text{(A1.12)}$$

and the state of the entangled pair can be written as

$$|\psi\rangle = \frac{1}{2^{3/2}}\left[|1'\rangle\left(\left(1+e^{i\phi}\right)|x'_1\rangle - \left(1-e^{i\phi}\right)|x'_2\rangle\right) - |2'\rangle\left(\left(1-e^{i\phi}\right)|x'_1\rangle - \left(1+e^{i\phi}\right)|x'_2\rangle\right)\right]. \qquad \text{(A1.13)}$$

Again we can calculate probabilities for separate detection of one or the other photon and for coincidences between a primary and a secondary counter:

$$\langle\psi|\mathbf{D'}_i \otimes \mathbf{I}_x|\psi\rangle = \langle\psi|\mathbf{I} \otimes \mathbf{d'}_j|\psi\rangle = \frac{1}{2} \qquad \text{(i,j = 1,2)} \qquad \text{(A1.14)}$$

$$\langle\psi|\mathbf{D'}_1 \otimes \mathbf{d'}_1|\psi\rangle = \langle\psi|\mathbf{D'}_2 \otimes \mathbf{d'}_2|\psi\rangle$$
$$= \frac{1}{8}\left(1+e^{-i\phi}\right)\left(1+e^{i\phi}\right) = \frac{1}{4}\left(1+\cos\phi\right) = \frac{1}{2}\cos^2\frac{\phi}{2} \qquad \text{(A1.15)}$$

$$\langle\psi|\mathbf{D'}_1 \otimes \mathbf{d'}_2|\psi\rangle = \langle\psi|\mathbf{D'}_2 \otimes \mathbf{d'}_1|\psi\rangle$$
$$= \frac{1}{8}\left(1-e^{-i\phi}\right)\left(1-e^{i\phi}\right) = \frac{1}{4}\left(1-\cos\phi\right) = \frac{1}{2}\sin^2\frac{\phi}{2}. \qquad \text{(A1.16)}$$

Interference reappears in the coincidences between a primary and a secondary detector. We might be tempted to say that this is due to the fact that inserting the beam splitter BS_3 destroys which-path information; but this is not entirely correct, as the absence of interference in (A1.14) shows. In 'measurement process language' we shall rather say that we are dealing with measurements of the type Q⊗R such that:

- in (A1.14) we have [R,M] = 0 in the first instance and [Q,O] = 0 in the second;
- in (A1.15) and (A1.16) we have both [Q,O] ≠ 0 and [R,M] ≠ 0, since the commutation relations between the **D'** and **D** observables and those between the **d'** and **d** observables are easily calculated by inverting equations (A1.2) and (A1.11):

$$\left[\mathbf{D'}_i, \mathbf{D}_j\right] = (-)^{\delta_{ij}}\, \frac{1}{2}\left(|1\rangle\langle 2| - |2\rangle\langle 1|\right)$$
$$\left[\mathbf{d'}_i, \mathbf{d}_j\right] = (-)^{\delta_{ij}}\, \frac{1}{2}\left(|x_1\rangle\langle x_2| - |x_2\rangle\langle x_1|\right) \qquad \text{(i,j = 1,2).} \qquad \text{(A1.17)}$$

Therefore the equivalence of the superposition (A1.8) and the corresponding mixture still holds for separate detection of one or the other photon, but not for coincidence counts. As we see, the 'measurement process language' offers here a surer guidance than the 'which-path information language.' But of course the absence of interference in (A1.14) can be understood also in terms of which-path information. We just need to notice that, when a measurement is performed on one photon only of the entangled pair, the which-path information is not actually lost, in spite of the other photon going through a beam splitter. The information is enfolded in the state of the other photon and is in principle recoverable by reversing the action of the beam-splitter.

Appendix 2

In the Hilbert space of a composite system the tensor product of projectors on pure states is the projector on the tensor product of the states.
If $\mathcal{H} = \mathcal{H}^{(S)} \otimes \mathcal{H}^{(\mathcal{M})}$, $\phi \in \mathcal{H}^{(S)}$, $\Phi \in \mathcal{H}^{(\mathcal{M})}$,

$$P_\phi \otimes P_\Phi = P_{\phi\otimes\Phi}\,. \tag{A2.1}$$

Proof:

$$\forall \chi_1, \chi_2 \in \mathcal{H}^{(S)}, \quad \forall \mathrm{X}_1, \mathrm{X}_2 \in \mathcal{H}^{(\mathcal{M})}:$$

$$\begin{aligned}
&\left(\chi_1 \otimes \mathrm{X}_1, P_\phi \otimes P_\Phi \chi_2 \otimes \mathrm{X}_2\right) \\
&= \left(\chi_1 \otimes \mathrm{X}_1, P_\phi \chi_2 \otimes P_\Phi \mathrm{X}_2\right) = \left(\chi_1, P_\phi \chi_2\right)\left(\mathrm{X}_1, P_\Phi \mathrm{X}_2\right) \\
&= \left(\chi_1, \phi\right)\left(\phi, \chi_2\right)\left(\mathrm{X}_1, \Phi\right)\left(\Phi, \mathrm{X}_2\right) = \left(\chi_1 \otimes \mathrm{X}_1, \phi \otimes \Phi\right)\left(\phi \otimes \Phi, \chi_2 \otimes \mathrm{X}_2\right) \\
&= \left(\chi_1 \otimes \mathrm{X}_1, P_{\phi\otimes\Phi} \chi_2 \otimes \mathrm{X}_2\right) \quad |||
\end{aligned} \tag{A2.2}$$

Appendix 3

The time-evolution of a density matrix which is a projector on a pure state is the projector on the time-evolution of the state:

$$\mathcal{U}_t P_\psi = U_t P_\psi U_t^{-1} = P_{U_t\psi} \tag{A3.1}$$

Proof:

$$\forall \chi_1, \chi_2 \in \mathcal{H}:$$
$$\begin{aligned}&\left(\chi_1, U_t P_\psi U_t^{-1} \chi_2\right)\\ &= \left(U_t^{-1}\chi_1, P_\psi U_t^{-1}\chi_2\right) = \left(U_t^{-1}\chi_1, \psi\right)\left(\psi, U_t^{-1}\chi_2\right)\\ &= \left(\chi_1, U_t\psi\right)\left(U_t\psi, \chi_2\right) = \left(\chi_1, P_{U_t\psi}\chi_2\right) \ \ |||\end{aligned} \tag{A3.2}$$

Appendix 4

Let M be a measurement recording observable ('pointer position' in an abstract sense) and {Q⊗R} the set of the observables of the composite system $\mathcal{S}+\mathcal{M}$ (object system and apparatus) that commute with M. Since we proved that for all such observables

$$\langle Q\otimes R\rangle = Tr(W_t Q\otimes R) = Tr(\overline{W_t} Q\otimes R), \tag{A4.1}$$

in order to verify the statement ($\mathcal{R}$) we only need to prove that

$$W_t^{(M)} = \sum_l P_l W_t P_l = \overline{W_t} \tag{A4.2}$$

where

$$W_t = \sum_k w_k P_{\sum_n c_n\phi_n\otimes\Phi_n(k)}, \tag{A4.3}$$

$$\overline{W_t} = \sum_{k,n} |c_n|^2 w_k P_{\phi_n\otimes\Phi_n(k)}, \tag{A4.4}$$

and

$$M = \sum_l m_l P_l \tag{A4.5}$$

is a spectral decomposition of M. We will not explicitly extend this result to a von Neumann chain of interacting systems, but such a generalization is straightforward and follows the same outline of the present proof.

Let $\{\psi_h\}$ be an arbitrary complete orthonormal set in $\mathcal{H}^{(S)}$ and $\{\Psi_{ij}\}$ a complete orthonormal set of eigenfunctions of M in $\mathcal{H}^{(\mathcal{M})}$ (with the first index specifying the eigenspace and the second being a degeneracy index):

$$M\Psi_{i,j} = m_i\Psi_{i,j}, \qquad P_l\Psi_{i,j} = \delta_{li}\Psi_{i,j} \tag{A4.6}$$

(furthermore, since the $\Phi_n(k)$ are eigenstates of M, $P_l\Phi_n(k) = \delta_{\ln}\Phi_n(k)$).

In order to prove (A4.2) we shall verify that

$$\left(\psi_h\otimes\Psi_{i,j}, W_t^{(M)}\psi_{h'}\otimes\Psi_{i',j'}\right) = \left(\psi_h\otimes\Psi_{i,j}, \overline{W_t}\psi_{h'}\otimes\Psi_{i',j'}\right), \quad \forall h,i,j,h',i.'j' \tag{A4.7}$$

Indeed,

$$\begin{aligned}
&\left(\psi_h\otimes\Psi_{i,j}, W_t^{(M)}\psi_{h'}\otimes\Psi_{i',j'}\right)\\
&= \sum_{k,l} w_k\left(\psi_h\otimes\Psi_{i,j}, P_l P_{\sum_n c_n\phi_n\otimes\Phi_n(k)} P_l\psi_{h'}\otimes\Psi_{i',j'}\right)\\
&= \sum_{k,l} \delta_{li}\delta_{li'} w_k\left(\psi_h\otimes\Psi_{i,j}, P_{\sum_n c_n\phi_n\otimes\Phi_n(k)}\psi_{h'}\otimes\Psi_{i',j'}\right)\\
&= \sum_{k} \delta_{ii'} w_k\left(\psi_h\otimes\Psi_{i,j}, P_{\sum_n c_n\phi_n\otimes\Phi_n(k)}\psi_{h'}\otimes\Psi_{i',j'}\right)\\
&= \sum_{k} \delta_{ii'} w_k\left(\psi_h\otimes\Psi_{i,j}, \sum_n c_n\phi_n\otimes\Phi_n(k)\right)\left(\sum_{n'} c_{n'}\phi_{n'}\otimes\Phi_{n'}(k), \psi_{h'}\otimes\Psi_{i',j'}\right)
\end{aligned}$$

$$= \sum_{k,n,n'} \delta_{ii'} c_n c_{n'}^{\ *} w_k \left(\psi_h \otimes \Psi_{i,j}, \phi_n \otimes \Phi_n(k)\right)\left(\phi_{n'} \otimes \Phi_{n'}(k), \psi_{h'} \otimes \Psi_{i',j'}\right)$$

$$= \sum_{k,n,n'} \delta_{ii'} c_n c_{n'}^{\ *} w_k \left(\psi_h, \phi_n\right)\left(\Psi_{i,j}, \Phi_n(k)\right)\left(\phi_{n'}, \psi_{h'}\right)\left(\Phi_{n'}(k), \Psi_{i',j'}\right) \qquad \text{(A4.8)}$$

Now the product $\delta_{ii'}\left(\Psi_{i,j}, \Phi_n(k)\right)\left(\Phi_{n'}(k), \Psi_{i',j'}\right)$ vanishes unless i = i' = n = n'; therefore:

$$\left(\psi_h \otimes \Psi_{i,j}, W_t^{(M)} \psi_{h'} \otimes \Psi_{i',j'}\right)$$

$$= \sum_{k,n} \delta_{ii'} \left|c_n\right|^2 w_k \left(\psi_h, \phi_n\right)\left(\Psi_{i,j}, \Phi_n(k)\right)\left(\phi_n, \psi_{h'}\right)\left(\Phi_n(k), \Psi_{i',j'}\right)$$

$$= \sum_{k,n} \delta_{ii'} \left|c_n\right|^2 w_k \left(\psi_h \otimes \Psi_{i,j}, \phi_n \otimes \Phi_n(k)\right)\left(\phi_n \otimes \Phi_n(k), \psi_{h'} \otimes \Psi_{i',j'}\right)$$

$$= \sum_{k,n} \delta_{ii'} \left|c_n\right|^2 w_k \left(\psi_h \otimes \Psi_{i,j}, P_{\phi_n \otimes \Phi_n(k)} \psi_{h'} \otimes \Psi_{i',j'}\right)$$

$$= \left(\psi_h \otimes \Psi_{i,j}, \overline{W_t} \psi_{h'} \otimes \Psi_{i',j'}\right) \ ||| \qquad \text{(A4.9)}$$

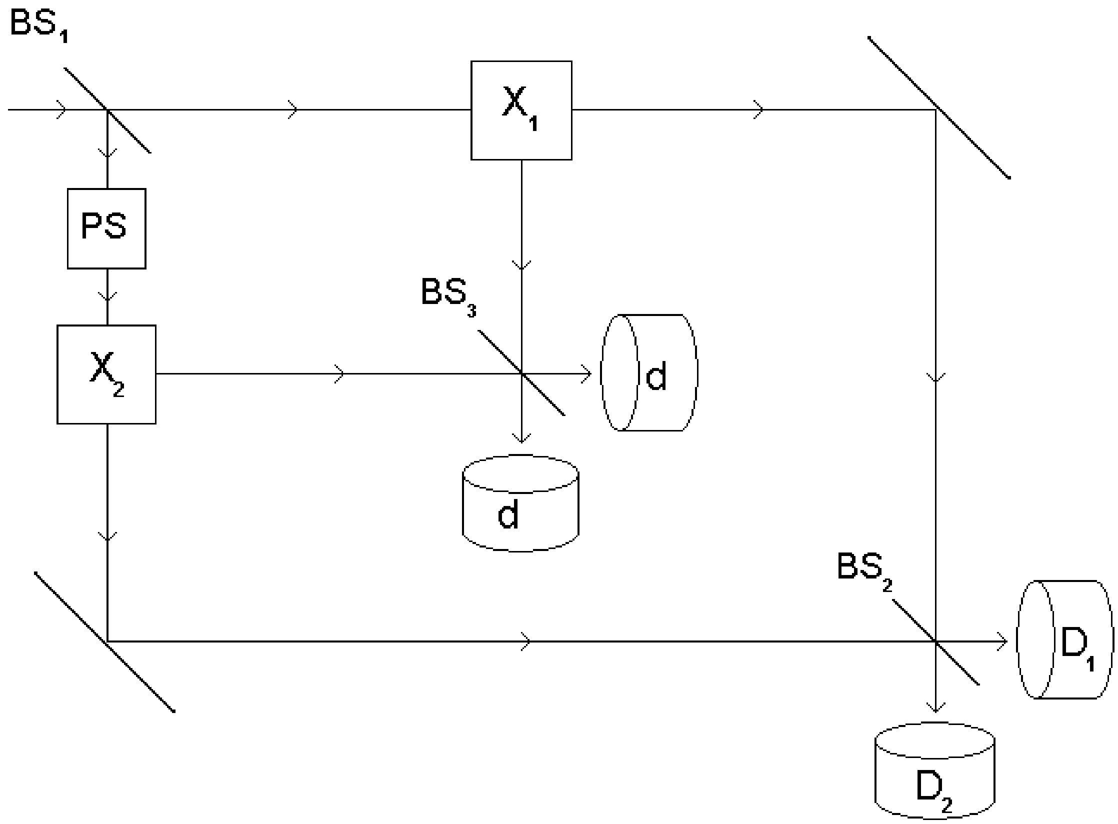

FIG. 1

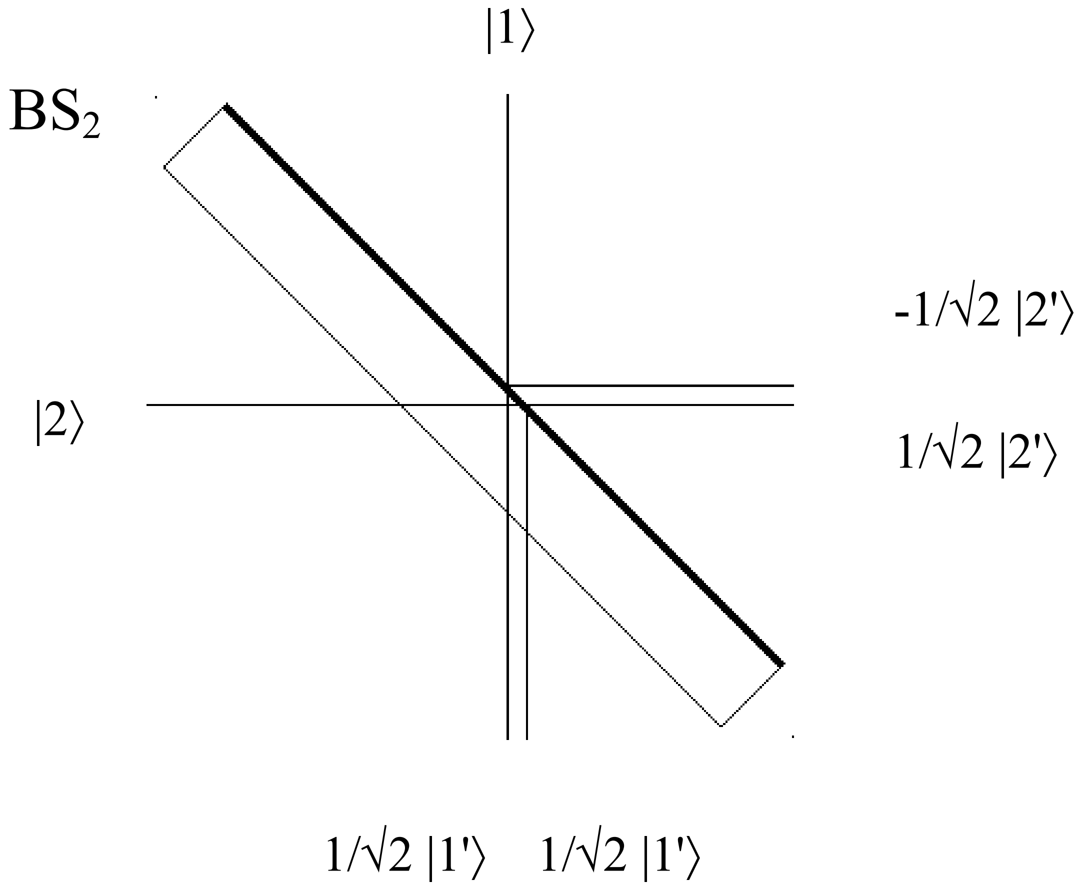

FIG. 2

Figure captions

FIG. 1. Schematic diagram of a quantum eraser experiment (from Greenstein and Zajonc, slightly modified). The experiment can be described in three steps. Step 1. A beam of photons enters a Mach-Zehnder interferometer and is split by a first beam splitter BS_1. A phase shifter PS alters the phase of one of the beams by ϕ and the two beams are re-combined by a second beam splitter BS_2. As ϕ is varied, an interference pattern is observerved in the counts of detectors D_1 and D_2. Step 2. Now the non-linear down-conversion crystal X_1 and X_2 are inserted, and the secondary photons generated by X_1 and X_2 are revealed by the additional detectors d_1 and d_2. The firing of d_1 or d_2 provides which-path information and the interference pattern disappears. Step 3. A third beam splitter BS_3 is placed at the intersection of the paths of the secondary photons from the down-conversion crystals. Now the firing of d_1 or d_2 no longer tells us which path the primary photon has taken, and an interference pattern re-appears in the coincidence count between a primary detector, D_1 or D_2, and a secondary detector, d_1 or d_2.

FIG. 2. At beam splitter BS_2 the partial waves corresponding to the states $| 1 \rangle$ and $| 2 \rangle$ are transmitted and reflected. The phase of the reflected part of $| 1 \rangle$ is shifted by π since it is reflected at the optically denser medium. The figure also applies to beam splitter BS_3, with $| x_1 \rangle$ and $| x_2 \rangle$ replacing $| 1 \rangle$ and $| 2 \rangle$.

Notes and references

[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] See e.g. E. Wigner, "Remarks on the Mind-Body Question", in *The Scientist Speculates*, ed. I.J. Good (Heinemann, London, 1961), 284.
[4] G.C. Ghirardi, A. Rimini, and T. Weber, Phys.Rev. D **34**, 470 (1986).
[5] See e.g. D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.O. Stamatescu, H.D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin and Heidelberg, 1996). Early work in this direction: A. Daneri, A. Loinger, and G.M. Prosperi, Nuclear Physics **33**, 297 (1962), reprinted in J,A. Wheeler and W.H. Zurek eds., *Quantum Theory and Measurement*, pp. 657-679 (Princeton University Press, 1983); L. Lanz, G.M. Prosperi, and A. Sabbadini, Nuovo Cimento **2B**, 184 (1971).
[6] The word 'immediately' in this context only reflects a practical concern. System and apparatus interact with their environment in such a way that correlations between their state and the state of the surrounding systems rapidly spread through the environment. An increasing portion of the environment 'records' the outcome of the measurement process as time goes on.
[7] T. Hellmuth, H. Walther, A. Zajonc and W. Schleich, *Phys. Rev. A* **35**, 2532 (1987).
[8] E. Jaynes, in *Foundations of Radiation Theory and Quantum Electronics*, ed. A.O. Barut (Plenum, New York, 1980), 37.
[9] M.O. Scully and K. Druhl, Phys. Rev. A **25**, 2208 (1982).
[10] See e.g. Z.Y. Ou, L.J. Wang, X.Y. Zou, and L. Mandel, Phys. Rev. A **41**, 566 (1990); P.G. Kwiat, A.M. Steinberg, and R.Y. Chiao, Phys. Rev. A **45**, 7729 (1992); T.J. Herzog, P.G. Kwiat, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **75**, 3034 (1995); T.B. Pittman, D.V. Strekalov, A. Migdall, M.H. Rubin, A.V. Sergienko, and Y.H. Shih, Phys. Rev. Lett. **77**, 1917-1920 (1996); P.D.D. Schwindt, P.G. Kwiat, and B.-G. Englert, Phys. Rev. A **60**, 4285 (1999); T. Tsegaye, G. Björk, M. Atatüre, A.V. Sergienko, B.E.A. Saleh, and M.C. Teich, Phys. Rev. A **62**, 032106 (2000); S.P. Walborn, M.O. Terra Cunha, S. Pádua, and C.H. Monken, Phys. Rev. A **65**, 033818 (2002); G. Teklemariam, E.M. Fortunato, M.A. Pravia, Y. Sharf, T.F. Havel, D.G. Cory, A. Bhattaharyya, and J. Hou, Phys. Rev. A **66**, 012309 (2002); H. Kim, J. Ko, and T. Kim, Phys. Rev. A **67**, 054102 (2003); U.L. Andersen, O. Glöckl, S. Lorenz, G. Leuchs, and R. Filip, Phys. Rev. Lett. **93**, 100403 (2004).
[11] G. Greenstein and A. Zajonc, *The Quantum Challenge* (Jones and Bartlett Publishers, Boston, 1997), 206.
[12] Z.Y. Ou, L.J. Wang, X.Y. Zou and L. Mandel, *Phys. Rev. A* **41**, 566 (1990).
[13] T. Hellmuth, H. Walther, A. Zajonc and W. Schleich, Phys. Rev. A **35**, 2532 (1987): Y.-H. Kim, R. Yu, S.P. Kulik, Y. Shih, and M.O. Scully, Phys. Rev. Lett. **84**, 1-5 (2000).
[14] The word "brain" in this context denotes a physical system whose states are correlated with the states of awareness of an observer. We don't need to deal with the complicated issue of what this system actually consists of. My only assumptions are:
(1) there is a physical system, which I call "brain" for short, such that the different perceptions or states of awareness of an observer correspond to different quantum states of this system;
(2) the quantum states corresponding to distinguishable perceptions or states of awareness are orthogonal to each other (e.g., the quantum states corresponding to "the observer is aware that the measurement has given the n^{th} result" and those corresponding to "the observer is aware that the measurement has given the m^{th} result", with $m \neq n$, are orthogonal to each other).
[15] Z.Y. Ou, L.J. Wang, X.Y. Zou and L. Mandel, *Phys. Rev. A* **41**, 566 (1990).